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The spin-up of liquid metal driven by a rotating magnetic field

By MARIUS UNGARISH†

Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

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The paper considers the flow field during spin-up from rest of liquid metal in a cylindrical stationary cavity due to a rotating transverse uniform low-frequency magnetic field. It is assumed that the Ekman and the magnetic Reynolds numbers are small. An approximate model, based on matching of Bödewadt-type layers with an inviscid core, with possible influence from the sidewall, for laminar flow is developed. It is shown that the angular velocity in the core is a function of time only. Analytical solutions for the angular velocity and the meridional flow in the core are presented, and supplemented by finite-difference results to show the sidewall effects. The spin-down following the switch-off of the magnetic forcing, the influence of the axial variations of the magnetic field, and the relevance to turbulent flows are discussed.

1. Introduction

Rotary electro-magnetic stirring is an important problem in the metal industry. Schematically, it can be described as follows: liquid metal in a stationary axisymmetric container (cavity) of finite length is subjected to the influence of a transverse uniform magnetic field that rotates about the axis. The objective is to describe the flow field of the liquid, which will evidently be of the 'rotating' or 'swirling' type.

Davidson (1992) gives a comprehensive list of references to studies on this problem and provides a 'unified theoretical framework', which is supported by comparison with experimental work of Robinson (1973). The analysis of Davidson is concerned with steady-state flows only, in which the motion is governed by a balance between the magnetic driving force and viscous friction on the boundaries.

The main objective of the present work is to investigate the initial transient stage of the flow field in the liquid metal from a state of rest (relative to the container) to the final steady state. This may be considered as a stage of 'spin-up from rest', i.e. from no swirl to the final state dominated by the rotational motion of the 'core'. (We shall also consider briefly the opposite situation of the time-dependent 'spin-down' flow that develops after the magnetic forcing is switched off, but this is a more conventional problem.) Davidson (1989) and Davidson & Boysan (1991) have treated transient flows in relevant circumstances, but have restricted their investigations to very short time periods, of less than one revolution of the fluid, for which the viscous boundary layers are not important. The transient to steady state which we study here is much longer, typically one hundred revolutions of the fluid, and critically affected by the viscous boundary layers.

Spin-up and spin-down of classic fluids is an extensively studied and fascinating

† Permanent address: Computer Science Department, Technion, Haifa 32000, Israel.

phenomenon. It is fairly well understood and modelled in axisymmetric (mostly cylindrical) configurations (Greenspan 1968; Benton 1973, Benton & Clark 1974; Weidman 1976*a*, *b*; Savaş 1992). However, the present spin-up is unusual because it is not driven by boundary conditions on the solid walls. Here the increase of angular momentum in the fluid is caused by a body force, the rotating magnetic field, while the stationary walls actually tend to retard the motion (i.e. to spin down the fluid). Consequently, features typical of both spin-up and spin-down to rest coexist in the present case, and the steady state reflects a balance between them.

The understanding of the initial spin-up stage is therefore interesting from an academic point of view. It has also obvious practical implications. For instance, in an aluminium casting process the spin-up time of the melted pool may affect the rate of replenishment and the speed of withdrawal of the solid ingot (see Davidson 1992, figure 2). Also, in a stirring process based on turning on and off the magnetic field for short time intervals (Kojima *et al.* 1983), the spin-up and spin-down phenomena have evident and perhaps major contributions.

The rotating flow investigated here may be classified as 'nonlinear' because the presence of the stationary solid walls is completely inconsistent with an attempt to consider the motion a 'linear' small perturbation around some state of solid-body rotation. A consequence is that a properly defined Rossby number (the ratio of inertia to Coriolis forces) would be of order unity.

To proceed, several simplifications are introduced. The driving body force is assumed to act in the azimuthal direction and modelled as time-independent, linearly increasing with the radial distance from the axis. The geometry of the container is taken to be cylindrical, with outer radius r_o^* and height H^* (the asterisk denotes a dimensional variable); the upper boundary is either solid or free. The aspect ratio, $H = H^*/r_o^*$, is of order unity. The molten metal is assumed a Newtonian incompressible fluid, and the viscosity forces are small compared with the dominant rotational (Coriolis) forces. Thus, the relevant Ekman number of the flow, $E = \nu^*/\Omega_{ref}^* r_o^{*2}$, is assumed very small, where ν^* is the kinematic viscosity coefficient and Ω_{ref}^* is the typical angular velocity of the fluid. For definiteness, Ω_{ref}^* is taken as the steady-state value at the centre, as discussed below. We assume that the flow is laminar.

The laminar-flow assumption is probably not a realistic one, but nevertheless it is expected to yield useful results. Since practically no experimental information is available for the initial stage of the flow we must rely on some extrapolations from related cases. First, in the experiments of Robinson (1973) the measured steady-state was turbulent. Second, a close analogy is expected between the present problem and the classic spin-down to rest in a cylinder, regarding the instantaneous behaviour of the boundary layers: in both cases a rotating core of fluid is matched to non-rotating solid boundaries by viscous mechanisms (the magnetic body force is relatively small in the viscous layer). There is evidence that in impulsive spin-down to rest the 'vertical' sidewall region is unstable and Taylor–Görtler vortices appear after a short time, essentially, during the first revolution of the fluid relative to the stationary wall for Ekman numbers smaller than 2.9×10^{-3}. For instance, Mathis & Neitzel (1985) report that the onset time was about $4\Omega_{ref}^{*-1}$ for $E \approx 6 \times 10^{-4}$ in a cylinder of aspect ratio $H = 18.7$. The onset time decreases as E decreases, and is not very sensitive to H. In addition, the 'horizontal' Bödewadt layers (also loosely referred to as Ekman layers) become unstable and turbulent for $E < 1.6 \times 10^{-3}$ (e.g. Savaş 1987; Lopez & Weidman 1996). It is therefore plausible to speculate that the magnetic-driven spun-up fluid will be laminar until the Ekman number based on the instantaneous angular velocity, $\Omega^*(t^*)$, reaches the critical value of about 2.9×10^{-3}. If the spin-up continues to higher values of Ω^* (smaller values of E, in particular $E < 1.6 \times 10^{-3}$) the flow is expected to become unstable and turbulent during the spin-up interval.

However, the above mentioned evidence on instability is also accompanied by the peculiar observation that *the mean angular flow in the core during spin-down is in fair agreement with the laminar prediction* (Weidman 1976*b*; Savaş 1992). This outcome may be attributed to the more basic experimental observations concerning the turbulent boundary layer of a rotating core on a stationary disk: the profiles of the mean velocity are similar to those of the classic Bödewadt laminar layer (Savaş 1987). Since the spin-down to rest of the core is dominated by the mean transport in these layers, the agreement between the laminar model and the mean turbulent case is less puzzling. The recent investigations of Lopez & Weidman (1996) also indicate the possibility of restabilization of the Bödewadt layer beneath a core of fluid in almost solid-body rotation. (We note in passing that the stability of the Bödewadt layer is still an intriguing topic, which has been actually investigated only in spin-down-to-rest circumstances; the flow discussed in this work provides, theoretically at least, novel conditions for producing and sustaining a Bödewadt layer.)

These considewrations lend the main justification to the present investigation. We show below that the spin-up of liquid metal driven by a magnetic field is controlled by the same boundary layers as the spin-down to rest of a regular fluid. The accepted fact that the laminar model gives qualitative and quantitative insights in the latter problem yields strong expectations that the same is valid in the former problem. We therefore proceed with the laminar theory, to provide at least a basis for comparison with experiments which, to the best of our knowledge, are not yet available.

An additional feature that we attempt to clarify in the present paper is the influence of the axial variation of the driving magnetic force on the resulting flow field during the spin-up stage. In the experiments of Robinson (1973) the magnetic azimuthal forcing decreased strongly with distance from the midplane (figure 8 in that paper). However, the measured angular velocity was almost constant in the axial direction. A similar behaviour was obtained in numerical experiments (see Davidson 1992, p. 683). Davidson, Short & Kinnear (1995) provide a theoretical interpretation to this curious result in the steady state by means of an order of magnitude analysis. A more formal incorporation of this feature is used below, and it is shown that it is consistent with the spin-up stage. Qualitatively, it reflects the inability of a fluid in which rotation is the dominant motion to support an axial pressure gradient in the (almost) inviscid core, in which the centrifugal accelerations are mainly supported by a strong radial pressure gradient. Quantitatively, the use of an axially averaged forcing for the evaluation of the angular velocity, as suggested by Davidson (1992), is also justified during the spin-up interval. However, the axial variation of the forcing about the average gives rise to some additional and quite strong motion in the meridional plane, which may be of importance to the stirring process.

2. Formulation

The container is a symmetric cylinder of radius r_o^* (which will serve as reference length). We use a cylindrical coordinate system, r, θ, z, with origin at the centre of the bottom solid wall; the top boundary at $z^* = H^*$ is either free or solid. For the sake of simplicity, the axial direction is also referred to as 'vertical' and the planes $z = \text{const.}$ as 'horizontal'. Again, the asterisk denotes a dimensional variable. Gravity effects are neglected.

We consider the driving force per unit volume of fluid to be of the form

$$\boldsymbol{F}^* = \tfrac{1}{2}\Omega_f^{*2} r^* \,[1 + f(z)]\, \hat{\theta}, \tag{1}$$

and

$$\int_0^H f(z)\mathrm{d}z = 0, \tag{2}$$

where Ω_f^* is a known constant. For definiteness, but without loss of generality, the driving force, and hence the resulting swirl, are assumed here in the positive $\hat{\theta}$-direction.

The connection between the model forcing (1) and a realistic magnetic forcing produced by a transverse magnetic field B^* rotating with frequency ω^* about the axis z of a contained metal fluid (of electrical conductivity and density σ^* and ρ^*) was pointed out, both theoretically and experimentally, by Davidson & Hunt (1987), Davidson (1992) and Davidson *et al.* (1995) as follows: if the magnetic Reynolds number μ^* and $\sigma^* U^* r_o^*$ is small, and the frequency ω^* is restricted to values smaller than $3/(\mu^*\sigma^* r_o^{*2})$, where μ^* is the permeability in free space and U^* the typical flow velocity, then the main resulting effect on the fluid can be approximated by (1) with

$$\Omega_f^* = B^*(\sigma^*\omega^*/\rho^*)^{1/2}.$$

In practical situations some deviations from the idealization (1) may occur, in particular a more complex dependency on r and perhaps an additional time-dependent oscillatory component. For the sake of simplicity of the analysis and results, we use (1) as an approximation and assume that these deviations will not cause qualitative differences from the present approach, and therefore the present results will provide at least significant insights and guidelines for both experiments and further theoretical investigations.

A typical value for the angular velocity is needed for the proper scaling of the Navier–Stokes equations. For this purpose we chose the steady-state angular velocity at the centre, which, as deduced from the analysis of Davidson (1992) and confirmed later by the present study, is

$$\Omega_{ref}^* = \left(\frac{1}{4c}\right)^{2/3} \Omega_f^* \left(\frac{\Omega_f^* H^{*2}}{\nu^*}\right)^{1/3} \tag{3}$$

where

$$c = 1.35\frac{n}{2}, \tag{4}$$

and n is the number of solid horizontal boundaries, i.e. $n = 1$ when the upper surface is free and $n = 2$ otherwise.

The dimensionless variables are scaled as follows: length with r_o^*, velocity with $\Omega_{ref}^* r_o^*$, pressure with $\rho^*\Omega_{ref}^{*2} r_o^{*2}$; the dimensionless time, denoted τ, is scaled with the spin-up time $H^*(\nu^*\Omega_{ref}^*)^{-1/2} = E^{-1/2}H/\Omega_{ref}^*$. Here

$$E = \frac{\nu^*}{\Omega_{ref}^* r_o^{*2}} \tag{5}$$

is the Ekman number.

The dimensionless equations of motion, on account of the driving force (1) are

$$\nabla \cdot \boldsymbol{v} = 0, \tag{6}$$

$$\frac{E^{1/2}}{H}\frac{\partial \boldsymbol{v}}{\partial \tau} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + E\nabla^2\boldsymbol{v} + \frac{E^{1/2}}{H}2cr\,[1+f(z)]\,\hat{\theta}, \tag{7}$$

and are subject to the initial condition $\boldsymbol{v} = 0$ and to the usual boundary conditions on solid walls and on the free surface when present.

When $E \ll 1$ the flow field can be modelled as a 'core' embedded in thin viscous boundary layers. In particular, at the 'horizontal' solid walls, Bödewadt-type viscous layers are expected to appear†. Since the $O(E^{1/2})$ circulation generated by these layers controls the interior flow, we anticipate (cf. Greenspan 1968, § 3.7), that in the core $0^+ < z < H^-$ the velocity field may be represented by an expansion in powers of $E^{1/2}$ according to

$$\boldsymbol{v} = E^{1/2}(u_1 + E^{1/2}u_2 + \cdots)\hat{r} + (v_0 + E^{1/2}v_1 + \cdots)\hat{\theta} + E^{1/2}(w_1 + E^{1/2}w_2 + \cdots)\hat{z}, \tag{8}$$

$$p = p_0 + E^{1/2}p_1 + \cdots, \tag{9}$$

$$\psi = E^{1/2}(\psi_1 + E^{1/2}\psi_2 + \cdots), \tag{10}$$

where the dependent variables v_0, u_1 etc., are expected to be $O(1)$ functions of r, z and τ, and the stream function is defined by $ur = \partial\psi/\partial z$, $wr = -\partial\psi/\partial r$. We substitute (8)–(9) in the governing equations and seek solutions for the zeroth- and first-order terms only; there is not sufficient resolution in the boundary-layer flow to validate results for additional terms.

The balance for the zeroth-order terms follows from the radial and axial momentum equation as

$$\frac{v_0^2}{r} = \frac{\partial p_0}{\partial r}, \quad \frac{\partial p_0}{\partial z} = 0, \tag{11}$$

from which $v_0 = v_0(r, \tau)$. The azimuthal momentum equation yields, to leading order, after some rearrangement,

$$\frac{\partial v_0}{\partial \tau} + Hu_1\frac{1}{r}\frac{\partial}{\partial r}(rv_0) = 2cr\,[1+f(z)]\,; \tag{12}$$

and the relevant continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_1) + \frac{\partial}{\partial z}w_1 = 0. \tag{13}$$

It is convenient to split the radial velocity into the axial-average and a perturbation about it components,

$$u_1(r,z,\tau) = \bar{u}_1(r,\tau) + \hat{u}_1(r,z,\tau) \tag{14}$$

such that $\int_0^H \hat{u}_1 \mathrm{d}z = 0$. Substitution of (14) in (12) yields

$$\frac{\partial v_0}{\partial \tau} + H\bar{u}_1\frac{1}{r}\frac{\partial}{\partial r}(rv_0) = 2cr, \tag{15}$$

and

$$\hat{u}_1 = \frac{2cr}{H(1/r)(\partial/\partial r)(rv_0)}f(z). \tag{16}$$

It is emphasized that, in view of (16) and (2), the net mass transport by the second component in the right-hand side of (14) is indeed zero. Therefore, considering the global continuity in a cylindrical control volume extending from bottom to top, we

† The magnetic forcing term turns out to be $O(E^{1/2})$ relative to the viscous terms in this layer, as indicated by Davidson (1992). Therefore, the standard results may be used.

obtain

$$\int_{0^+}^{H^-} u_1 \mathrm{d}z = H\bar{u}_1(r,\tau) = -\left[\tilde{Q}^T + \tilde{Q}^B\right] E^{-1/2}, \tag{17}$$

where $\tilde{Q}$ is the contribution of one boundary layer to radial volume transport (the integral of the radial velocity over the width of the layer) and the superscripts T and B denote top and bottom.

If the horizontal boundary is a solid wall the layer on it is of the Bödewadt type. The time of formation and adjustment of such layers is of order Ω_{ref}^{*-1}, hence they are quasi-steady on the spin-up time interval of order $E^{-1/2}\Omega_{ref}^{*-1}$. Consequently, we can employ the Ekman layer transport correlation†

$$\begin{aligned} \tilde{Q} &= -\tfrac{1}{2}\kappa E^{1/2}(v_0/r)^{1/2} r \text{ on a solid wall} \\ &= 0 \quad \text{on a free surface,} \end{aligned} \tag{18}$$

where

$$\kappa \approx 1.35.$$

Substitution of (18) into (17) gives

$$H\bar{u}_1(r,\tau) = \frac{n}{2}\kappa(v_0 r)^{1/2}, \tag{19}$$

where $n = 1$ if the upper boundary is free or $n = 2$ if the upper boundary is rigid. Now we can eliminate $H\bar{u}_1$ from (15) and (19) to obtain a single equation for the azimuthal velocity. It is convenient to first recast this result of (15) and (19) into an equation for the angular momentum,

$$\frac{\partial \Gamma}{\partial \tau} + \tfrac{1}{2}n\kappa\Gamma^{1/2}\frac{\partial \Gamma}{\partial r} = 2cr^2, \quad \text{where } \Gamma = v_0 r; \tag{20}$$

initially, $\Gamma = 0$. This is a standard equation amenable to solution by the method of characteristics, and we observe that all the characteristics satisfy $\mathrm{d}r/\mathrm{d}\tau > 0$ for $\tau, r > 0$, and $\mathrm{d}r/\mathrm{d}\tau = 0$ at $r = 0$. Hence no boundary conditions on the lateral wall, $r = 1$, are required or possible; we shall see later that such conditions can be incorporated with the addition of a viscous term into the balance of azimuthal momentum, but the influence is confined to a thin region if $HE^{1/2}$ is small. Regarding the direction of propagation of the characteristics of the equation for Γ in the core, the present problem is identical with that of the spin-down to rest in a cylinder, but is different from the spin-up from rest in a cylinder. Indeed, in the latter problem it is the inward propagation of the characteristics from $r = 1$ that carries the essential information.

With these observations in mind, we can facilitate the solution of the azimuthal motion in the core by considering the equation for the angular velocity, readily obtained from (20) as

$$\frac{\partial \Omega}{\partial \tau} + n\kappa\Omega^{3/2} + \underline{\tfrac{1}{3}n\kappa r\frac{\partial \Omega^{3/2}}{\partial r}} = 2c, \quad \text{where } \Omega = v_0/r = \Gamma/r^2, \tag{21}$$

subject to $\Omega(r, \tau = 0) = 0$. It is evident that, if no boundary condition on $r = 1$ is to be imposed, an initial r-independent Ω will remain r-independent, i.e. the underlined

† The dimensional thickness of this layer is $\sim 8[\nu^*/\Omega_I^*]^{1/2}$. In dimensional form, $\tilde{Q}^* = -0.5\kappa[\nu^*/\Omega_I^*]^{1/2}\Omega_I^* r^*$, where Ω_I^* is the angular velocity (in absolute value) of the fluid far away from the stationary disk; the $-$ sign shows that the transport is from the periphery to the centre. See Greenspan (1968, § 3.2).

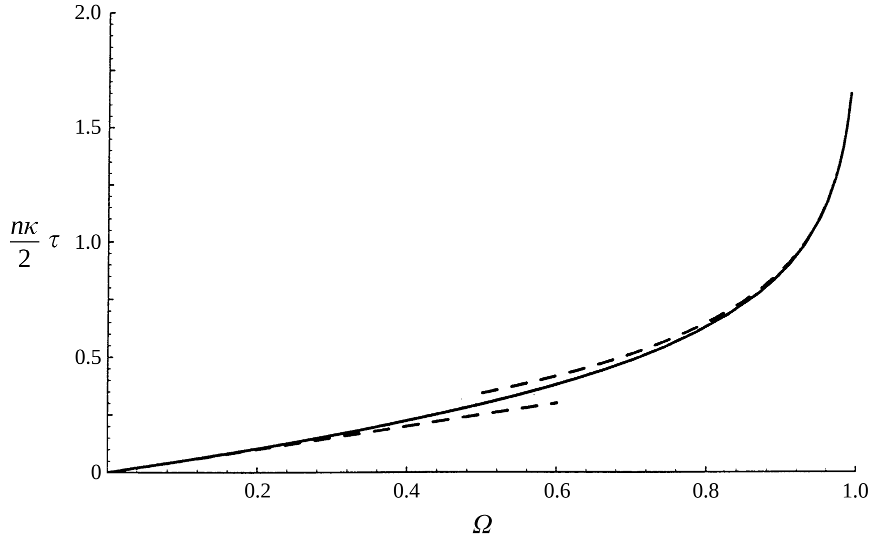


FIGURE 1. Reduced time vs. Ω, inviscid core: —, exact and - - -, approximate results of (21).

term on the left-hand side starts as and remains identically zero. (It will be shown later that even the influence of the sidewall is unable to excite the $\partial\Omega^{3/2}/\partial r$ term outside the viscous shear layer.) We conclude that in the core under consideration the angular velocity is a function of time only, that we determine as follows.

First, we note that setting the coefficient c in the definition (3) to

$$c = \frac{n\kappa}{2} \approx 1.35\frac{n}{2}$$

renders $\Omega = 1$ in steady state, which justifies (3)–(4). With this value of c, the solution $\Omega(\tau)$ of (21) can be expressed in implicit form as

$$\tau = \frac{2}{n\kappa}\left[\frac{1}{3}\ln\frac{(1+b+b^2)^{1/2}}{|1-b|} + \frac{1}{\sqrt{3}}\left(\arctan\frac{\sqrt{3}b}{2+b} - \frac{\pi}{3}\right)\right], \quad \text{where } b = \Omega^{-1/2}. \quad (22)$$

Series expansions provide the explicit approximations

$$\Omega \approx n\kappa\tau \quad \text{for} \quad \tau \ll 1, \quad (23)$$

$$\Omega \approx 1 - 2\sqrt{3}\exp\left[-\left(\frac{\pi}{2\sqrt{3}} + \tfrac{3}{2}n\kappa\tau\right)\right], \text{ for } \tau \gg \tfrac{2}{3}. \quad (24)$$

Figure 1 displays these results. The approximations (23) and (24) have fair accuracy for $\frac{1}{2}n\kappa\tau$ smaller and larger than 0.5, respectively.

Essentially, 99% of the steady-state swirl is achieved in $\tau \approx 1.7(2/n)$ dimensionless time units. In dimensional form the corresponding time is

$$t^* \approx 1.7(2/n)^{2/3}H^* \Bigg/ \left[0.32\nu^*\Omega_f^*\left(\frac{\Omega_f^* H^{*2}}{\nu^*}\right)^{1/3}\right]^{1/2}.$$

Thus, if the upper boundary is free, the spin-up time is longer than in the configuration with two rigid walls. The factor of increase is only $2^{2/3}$ because in the steady state

Ω^*_{ref} itself is larger in the former case, which partly compensates for the lack of the upper layer during the spin-up process.

We remark about (21) with $\Omega(r, 0) = 0$ that

(1) The influence of the outer wall, $r = 1$, or any other radial dependency, is missing. Thus, this equation also describes an infinite disk configuration. In the steady-state limit the analysis of Davidson (1992, § 3) for such a configuration is recovered. In classical spin-up from rest (Wedemeyer 1964) the outer wall and the radial dependency are essential. On the other hand, in spin-down to rest (Weidman 1976) the interior flow is not dependent on r, and is given by a similar equation (with zero right-hand side and a non-zero initial condition). Thus, although we deal here with a spin-up-from-rest problem, we detect close similarity with the spin-down-to-rest problem.

(2) The second term on the left-hand side represents the boundary layer effect. At the beginning of the motion Ω grows due to the magnetic forcing (the right-hand side) with little hindering from the boundary layer friction, as reproduced by (23).

(3) Reconsidering the expansions (8)–(9), we realize that the validity is restricted to small values of the *instantaneous* Ekman number, $[E/\Omega(\tau)]^{1/2} \ll 1$, which on account of (23) can be expressed as

$$\tau \gg E. \tag{25}$$

This is actually a mild restriction if we follow the development of the flow field for many revolutions of the fluid as in a typical spin-up process. According to (23) the time required for the fluid to perform the first revolution is $\tau \sim E^{1/4}$, and the poor accuracy of the results during the first cycle is common to approximations based on quasi-steady Ekman layers. Indeed, for times shorter than one revolution of the fluid an inviscid analysis as presented by Davidson (1989) and Davidson & Boysan (1991) may be applied.

With the known solution $v_0 = \Omega(\tau)r$ the meridional flow can be easily determined. First, using (14), (16)–(18), we express the radial velocity as

$$u_1(r, z, \tau) = \frac{n\kappa}{2H}\left[[\Omega(\tau)]^{1/2} + \frac{1}{\Omega(\tau)}f(z)\right]r; \tag{26}$$

next, using the continuity equation and in conjunction with (10) gives

$$w_1(r, z, \tau) = -\frac{n\kappa}{H}\left[[\Omega(\tau)]^{1/2}\, z + \frac{1}{\Omega(\tau)}\int_0^z f(\bar{z})\mathrm{d}\bar{z}\right] + \kappa[\Omega(\tau)]^{1/2}\,, \tag{27}$$

$$\psi_1 = -\tfrac{1}{2}r^2 w_1. \tag{28}$$

The last term in (27) is the contribution of the viscous layer on the bottom. Recall that $f(z)$ represents the variation of the magnetic driving. This supports the axial variation of the azimuthal Coriolis acceleration and allows the formation of a z-dependent component of the radial velocity. This component may be large at small τ because when the swirl velocity is small there is little Coriolis hindrance to the radial motion. But evidently, for $\tau \to 0$, if $f(z)$ is not identically zero equations (26)–(27) yield a non-physical result. This is because the instantaneous Ekman number is not small near the onset of the motion and the expansion (8) is not valid. Actually, the condition (25) – or the realization that the analysis should not be applied during the first revolution of the fluid – is sufficient to eliminate the non-physical behaviour of (26) for $\tau \to 0$. The formal validity of the expansion procedure requires $u_1 = O(1)$ which implies $|f(z)| \leqslant E^{1/4}$ for the times of interest, but a violation of this formal restriction may be tolerated because Ω is substantially larger than $E^{1/2}u_1$ even for $|f(z)| = O(1)$, and this is the backbone of the simplifications used here, in particular of (11). Actually, it can be argued using order of magnitude considerations that, even during the very initial stage the swirl motion is larger than the meridional circulation, and that the latter always tends to smooth out z-variations in the former.

The influence of the axial variation of the driving force on the meridional flow in the core is illustrated in figure 2 for representative forms of $f(z)$: (*a*) zero, (*b*) linear and (*c*) parabolic profiles. It is worth noting that non-zero $f(z)$ creates regions of negative u which may be beneficial in stirring processes.

2.1. *Outer wall influence*

The core angular velocity $\Omega(\tau)$ as given by (22) must be accommodated to the no-slip boundary condition at $r = 1$. The similarity with the linear spin-down problem suggests that a viscous region appears near the cylindrical wall, reminiscent of the $E^{1/4}$ Stewartson layer, where v_0 varies strongly with r, but is still independent of z. Indeed, an order of magnitude analysis of the azimuthal component of (7) shows that the viscous term $E(\partial^2 v_0/\partial r^2)$ is able to counteract the forcing term $(E^{1/2}/H)2c$ in a boundary layer of thickness $H^{1/2}E^{1/4}$. On the other hand, the convection of angular momentum into this layer is expected to reduce its thickness and invalidate the z-independence of v_0.

For a more quantitative modelling of this region we may add the pertinent shear term, $HE^{1/2}(\partial/\partial r)(1/r)(\partial/\partial r)rv_0$, to the right-hand side of (12) and (15). Therefore the equation for $\Omega(r, \tau)$, that replaces (21), is

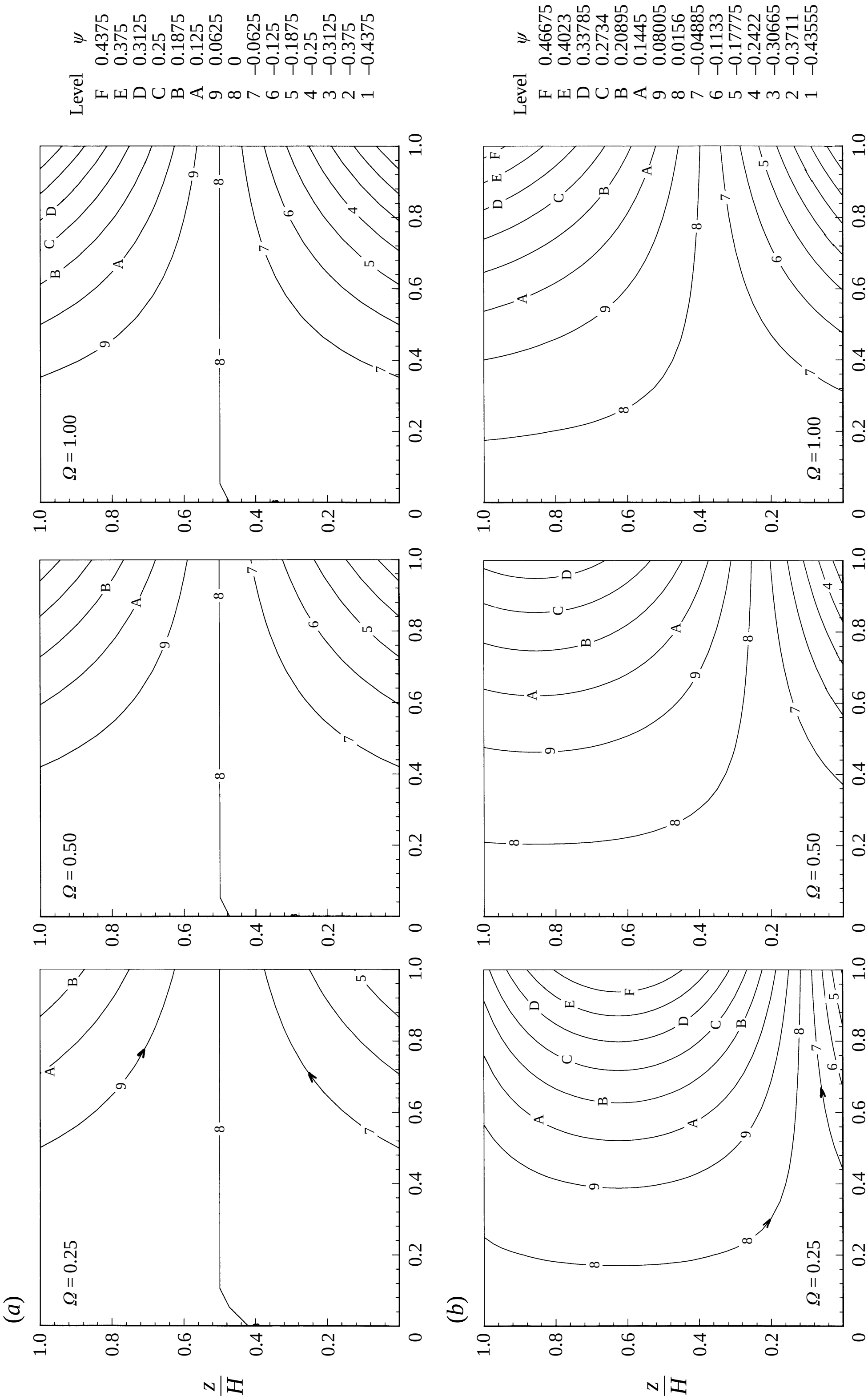
$$\frac{\partial \Omega}{\partial \tau} + n\kappa\Omega^{3/2} + \underline{\tfrac{1}{3}n\kappa r\frac{\partial \Omega^{3/2}}{\partial r}} = 2c + HE^{1/2}\left(\frac{\partial^2\Omega}{\partial r^2} + \frac{3}{r}\frac{\partial \Omega}{\partial r}\right), \tag{29}$$

subject to $\Omega(r, 0) = 0, \Omega(1, \tau) = 0$, and $\partial\Omega/\partial r = 0$ at $r = 0$.

The solution is readily obtained by a finite-difference method, and typical results are shown in figure 3. As expected, the flow in the interior $r < 1 - H^{1/2}E^{1/4}$ is accurately described by the 'inviscid core' $\Omega = \Omega(\tau)$ result, see (22). Although the underlined convection term is now excited (unlike in the treatment of the core), its influence remains restricted to the shear layer region and the angular velocity in the core remains r-independent. Actually, this convection term causes a considerable shrinking of the shear layer (a known effect for the $E^{1/4}$ layer in a sink region for non-small Rossby numbers). Indeed, the dominant balance between the convection term $(n\kappa/3)(\partial\Omega^{3/2}/\partial r)$ and the shear term $HE^{1/2}(\partial^2\Omega/\partial r^2)$ would even indicate a layer of thickness $O(E^{1/2})$. A thin layer with strong shear action is needed to reduce the angular momentum of the fluid convected from the core into the side layer (by about 20%) in order to make it compatible with the fluid returned toward the centre by the Bödewadt layers. However, the details of the flow in the sidewall layer and of its matching to the Bödewadt layer is beyond the scope of this paper (for some indications on the possible relevant features see Greenspan 1968, § 3.3).

An essential difference between the classic problem of spin-down to rest in the absence of a magnetic field case and the present problem regarding the viscous sidewall layer must be noted: in spin-down to rest the viscous region diffuses continuously into the interior but here the thickness of the side layer increases to a final, fixed steady-state value. In the former case the effective Ekman number increases with time, while in the latter it decreases to a steady-state value.

However, it is well known that profiles of Ω vs. r of the form given in figure 3 are prone to Taylor–Görtler instabilities, hence the results for this region should be


(a)
(b)
Ω = 0.25
Ω = 0.50
Ω = 1.00
z/H
Level ψ
F 0.4375
E 0.375
D 0.3125
C 0.25
B 0.1875
A 0.125
9 0.0625
8 0
7 −0.0625
6 −0.125
5 −0.1875
4 −0.25
3 −0.3125
2 −0.375
1 −0.4375
Level ψ
F 0.46675
E 0.4023
D 0.33785
C 0.2734
B 0.20895
A 0.1445
9 0.08005
8 0.0156
7 −0.04885
6 −0.1133
5 −0.17775
4 −0.2422
3 −0.30665
2 −0.3711
1 −0.43555

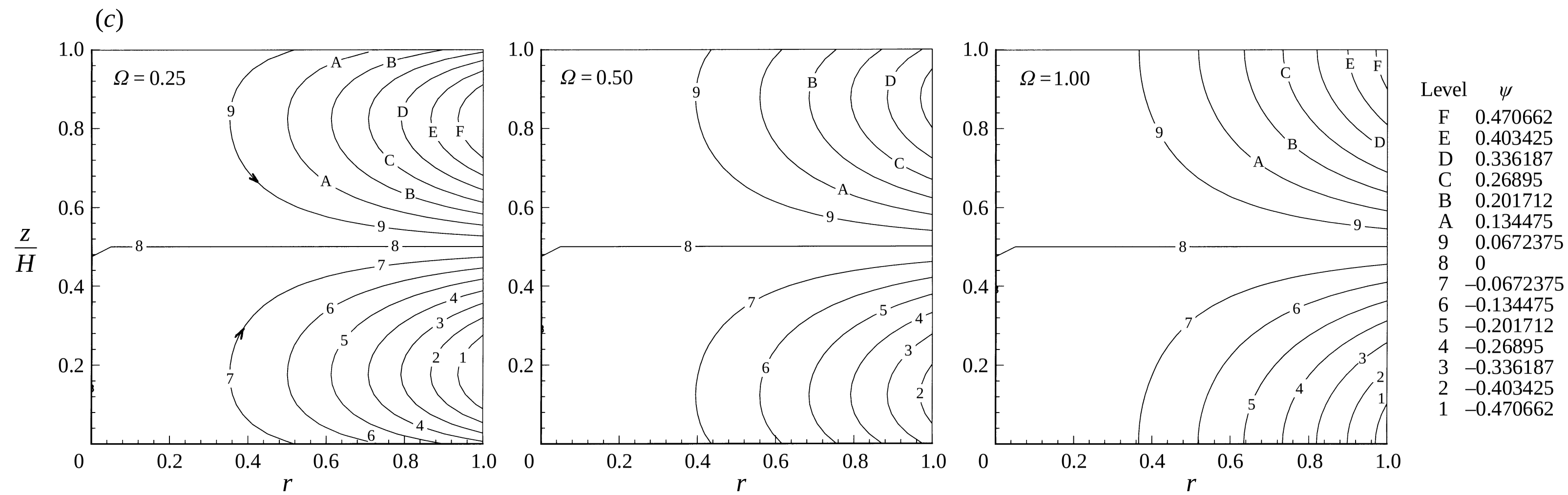


FIGURE 1 Stream function ψ_1/κ in the core of a configuration with solid top and bottom walls ($n = 2$), during spin-up at times corresponding to $\Omega = 0.25, 0.5$ and 1 ($\tau\kappa \approx 0.13, 0.30, 1.7$). The axial variation of the forcing is: (*a*) $f(z) = 0$, (*b*) $f(z) = 0.5(1 - 2z/H)$, (*c*) $f(z) = 0.5[1 - 3(1 - 2z/H)^2]$.

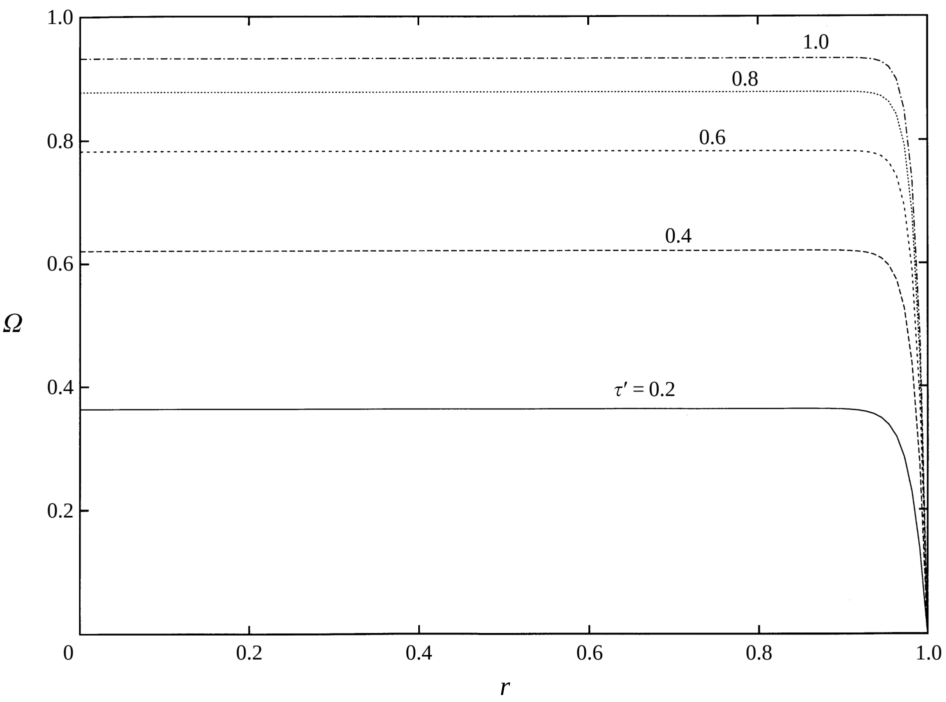


FIGURE 2. Ω vs. r (in core and near sidewall), for various τ' ($= \tau\kappa$), upper and lower walls rigid, $(HE^{1/2}/\kappa) = 10^{-2}$. Spin-up from rest

considered mainly as a qualitative indication on the pattern by which the internal swirl is accommodated to the wall; the quantitative effects of instabilities and turbulence on the foregoing side layer considerations require a separate investigation. Mathis & Neitzel (1985) report that in their spin-down-to-rest experiments the onset time for Taylor–Görtler vortices increases with E. In particular: (*a*) for $E \approx 6 \times 10^{-4}$ the onset occurs in less than one revolution of the fluid, considerably less than the spin-up process, and (*b*) no Taylor–Görtler vortices were observed when $E > 2.9 \times 10^{-3}$. We may infer that the same results are valid for the present flow during the spin-up interval, because the effective Ekman number is $E/\Omega(\tau)$, which is of course larger than E during the spin-up stage.

3. Spin-down

Suppose that at some advanced stage of the previously discussed process, say at $\tau = 1$, the magnetic field is turned off. The subsequent spin-down motion may be of interest in two practical circumstances:

(*a*) when the relevant region of molten metal is withdrawn from the magnetic section by the axial motion of the ingot;

(*b*) when the stirring process is designed as a succession of on-off magnetic forcing, with possible change of direction (Kojima et al. 1983).

According to the previous results, the spin-down will start from a state of almost uniform angular velocity. Therefore, the fluid is expected to behave like in the classic cases of spin-down to rest (Weidman 1976*a*, *b*; Savaş 1992).

From the viewpoint of mathematical modelling, we may use the foregoing formulation with no forcing term; in particular, to obtain Ω during spin-down we set the

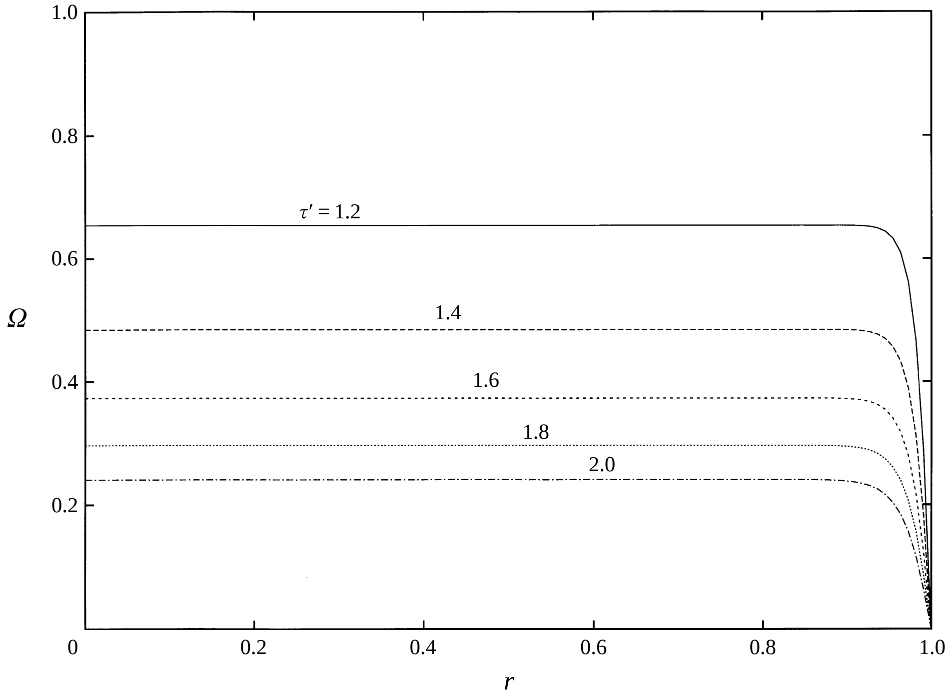


FIGURE 3. Spin-down in the configuration of figure 3, with magnetic forcing turned off at $\tau' = 1$.

first term on the right-hand side of (21) and (29) equal to zero, and change the 'initial conditions'. A typical behaviour is shown in figure 4. Comparison with figure 3 shows that the main spin-down events are qualitatively similar, but with time reversed, to the features of the spin-up under the magnetic driving: the angular velocity is almost uniform and dominant; the Ekman layers pump fluid from the periphery into the interior; and the net radial flow in the core is toward the periphery. The evident difference between spin-up and spin-down here is the increase vs. decrease of the angular velocity in the core and the compression vs. expansion of the sidewall layer. Of course, there may be a meridional flow in spin-up due the the axial variation of the driving force, $f(z) \not\equiv 0$, that will not be present in spin-down after the removal of the driving force.

Davidson (1989) and Davidson & Boysan (1991) discuss a flow of metal fluid in a cylinder that may appear after the removal of the magnetic field. An essential assumption in these papers is that the swirl velocity is z-dependent and the meridional circulation is strong (i.e. the velocity components u, v, w are of the same order of magnitude). In the present flow field that emerged from the spin-up from rest the axial dependency of the swirl and the meridional circulation are expected to be small, $O(E^{1/2})$. Hence the type of flow studied by Davidson (1989) and Davidson & Boysan (1991) is not expected to be an essential component in the spin-down stage of the present configuration.

4. Concluding remarks

A simple model for the spin-up from rest of liquid metal in a cylindrical cavity due to a rotating magnetic field has been presented. The solution provides insight into the behaviour of the velocity field and furnishes the time span of the process. The driving force was assumed to vary linearly with r, but moderate and smooth deviations from this form, as expected in practical devices, are not expected to cause significant differences from the present results.

We found that the spin-up behaviour of a magnetic-driven liquid is very different from the classic case studied by Wedemeyer (1964), because the forcing is different. In Wedemeyer's problem the spin-up effects in the core propagate from the periphery into the interior: the angular velocity of the fluid increases with r in the domain $r > r_F(\tau)$ and is zero in the shrinking region $r < r_F(\tau)$, where $r_F(\tau) = \exp(-2\tau/n)$. In the present case the angular velocity in the core is not r-dependent, and the radial motion in the core is toward the periphery. These features are more typical of the classic spin-down-to-rest process studied by Weidman (1976*a*, *b*); we therefore expect that the flow becomes unstable and turbulent before the accomplishment of the process, see also Savaş (1987) and Lopez & Weidman (1996).

The laminar flow analysis performed here is expected to be relevant to the mean motion. The verification of this conjecture by additional theoretical (perhaps numerical) work is a topic for future investigation. We also hope that the present results will serve as guidelines for the very difficult experimental investigation of the spin-up and spin-down of liquid metal.

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